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WEAK CO-T-COFIBRATIONS AND HOMOLOGY DECOMPOSITIONS

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Abstract. In this paper, we define a concept of weak co-T-cofibration which is a generalization of weak H'-cofibration, and study some properties of weak co-T-cofibration and relations between the weak co-T-cofibration and the homology decomposition for a cofibration.

1. Introduction

Eckmann and Hilton [2, 4] introduced a homology decomposition of 1-connected polyhedron as a dual concept of Postnikov system. Moreover, Eckmann and Hilton [3, 4] and Moore [5] introduced, as a generalization of this notion, the notion of the homology decomposition of a map. On the other hand, Tsuchida [7] introduced, as an intermediate notion of the above two decompositions, a notion of the homology decomposition for a cofibration $B \xrightarrow{q} X \xrightarrow{p} F$. If B reduces to a point, then such a decomposition reduces to the usual homology decomposition for X. Tsuchida [7] introduced the notion of weak H'-cofibration as a generalization of the induced cofibration or H'-cofibration defined in [6], and studied the relations between the weak H'-cofibration and the homology decomposition for a cofibration.

In this paper, we define a concept of weak co-T-cofibration which is a generalization of weak H'-cofibration, and study some properties of weak co-T-cofibration and relations between the weak co-T-cofibration and the homology decomposition for a cofibration. If an (inclusion) cofibration $B \xrightarrow{q} X \xrightarrow{p} F$ is obtained by applying the suspension functor Σ to an (inclusion) cofibration $B' \xrightarrow{q'} X' \xrightarrow{p'} F'$, then we obtain that

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there is the homology decomposition $\{X_n, F_n, i_n, j_n, p_n, q_n\}$ for $B \xrightarrow{q} X \xrightarrow{p} F$ such that $B \xrightarrow{q_n} X_n \xrightarrow{p_n} F_n$ is a weak co-T-cofibration for each n. Moreover, if $B \xrightarrow{q} X \xrightarrow{p} F$ is an weak co-T-cofibration and Y is a co-T-space with co-T-structure μ and $f : Y \to X$ is a coprimitive, and $X \xrightarrow{i} C_f \xrightarrow{\pi} \Sigma Y$ is an induced cofibration via f, then we obtain that $B \xrightarrow{iq} C_f \xrightarrow{\bar{p}} C_{pf}$ is a weak co-T-cofibration.

Throughout this paper, space means a space of the homotopy type of 1-connected locally finite CW complex. We assume also that spaces have non-degenerate base points. All maps shall mean continuous functions. All homotopies and maps are to respect base points. The base point as well as the constant map will be denoted by *. For simplicity, we use the same symbol for a map and its homotopy class. Also, we denote by $[X, Y]$ the set of homotopy classes of pointed maps $X \to Y$. The identity map of space will be denoted by 1 when it is clear from the context. The diagonal map $\Delta\colon X \to X \times X$ is given by $\Delta(x) = (x, x)$ for each $x \in X$, the folding map $\nabla\colon X \vee X \to X$ is given by $\nabla(x, *) = \nabla(*, x) = x$ for each $x \in X$. ΣX denote the reduced suspension of X and ΩX denote the based loop space of X. The adjoint functor from the group $[\Sigma X, Y]$ to the group $[X, \Omega Y]$ will be denoted by τ. The symbols e and e' denote $\tau^{-1}(1_{\Omega X})$ and $\tau(1_{\Sigma X})$ respectively.

2. Weak co-T-cofibratons

Let $B \xrightarrow{q} X \xrightarrow{p} F$ be a cofibration and let $f : Y \to X$ be a map. Let C_f (resp. C_{pf}) denotes the space obtained by attaching the reduced cone, cY, over Y to X (resp. F) by means of f (resp. pf), i.e. $C_f = cY \cup_f X$ (resp. $C_{pf} = cY \cup_{pf} F$). Then $F \xrightarrow{s} C_{pf} \to \Sigma Y$ is an inclusion cofibration and the following diagram is commutative;

$$\begin{array}{ccccc} Y & \xrightarrow{f} & X & \xrightarrow{p} & F \\ {\scriptstyle\iota}\downarrow & & {\scriptstyle i}\downarrow & & {\scriptstyle s}\downarrow \\ cY & \xrightarrow{k} & C_f & \xrightarrow{\bar{p}} & C_{pf}, \end{array}$$

where ι, k, i and s are inclusion maps and $\bar{p}$ is defined by $\bar{p}([y,t]) = [y,t]$, $[y,t] \in cY$ and $\bar{p}(x) = p(x)$, $x \in X$. Since $\bar{p}([y,1]) = [y,1] = pf(y)$ and $\bar{p}(fy) = p(fy)$, $\bar{p}$ is well defined. Thus it is obtained [7] that $B \xrightarrow{iq} C_f \xrightarrow{\bar{p}} C_{pf}$ is a cofibration. *A homology decomposition for* *a cofibration* $B \xrightarrow{q} X \xrightarrow{p} F$ consists of a sequence of spaces and maps $(X_n, F_n, i_n, j_n, q_n, p_n)$ satisfying

(I) $X_1 = B, j_1 = q, q_1 = 1$.

(II) $B \xrightarrow{q_n} X_n \xrightarrow{p_n} F_n$ is an inclusion cofibration.

(III) $q = j_n q_n : B \xrightarrow{q_n} X_n \xrightarrow{j_n} X$.

(IV) $X_{n-1} \xrightarrow{i_n} X_n \to K(H_n(F), n)$ is an inclusion cofibration ($n \geq 2$), where $i_2 = q_2$.

(V) maps q_n, j_n induce the following; (1) $(j_n)_* : H_r(X_n) \xrightarrow{\cong} H_r(X)$ for $r < n$, (2) In the sequence $H_n(B) \xrightarrow{q_{n*}} H_n(X_n) \xrightarrow{j_{n*}} H_n(X)$, q_{n*} is a monomorphism, j_{n*} is an epimorphism and Im $q_{n*} \supset$ Ker j_{n*}, (3) $q_{n*} : H_r(B) \xrightarrow{\cong} H_r(X_n)$ for $r > n$.

(VI) (1) A map $\bar{j_n} : F_n \to F$ induced by j induces $\bar{j_{n*}} : H_r(F_n) \xrightarrow{\cong} H_r(F)$ for $r \leq n$, (2)$H_r(F) = 0$ for $r > n$.

It is known [7] that there exists a homology decomposition for an inclusion cofibration $B \xrightarrow{q} X \xrightarrow{p} F$.

A cofibration $B \xrightarrow{q} X \xrightarrow{p} F$ is called a *weak H'-cofibration* [7] if there exists a map $\theta : X \to F \vee X$ and a homotopy $H_t : X \to F \times X$ such that

$$(a)\quad \begin{array}{ccc} B & \xrightarrow{i_2} & F \vee B \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle (1 \vee q)} \\ X & \xrightarrow{\theta} & F \vee X, \end{array}$$

is homotopy commutative, where i_2 is the injection into the second factor. $(b) H_0 = j\theta$ and $H_1 = (p \times 1)\Delta$, where $j : F \vee X \to F \times X$ is the inclusion.

Let Y be a co-H-space with co-H-structure $\mu : Y \to Y \vee Y$ and let $B \xrightarrow{q} X \xrightarrow{p} F$ be a weak H'-cofibration. Then a map $f : Y \to X$ is called [7] to be *coprimitive with respect to co-H-structure* μ if

$$\begin{array}{ccc} Y & \xrightarrow{\mu} & Y \vee Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (pf \vee f)} \\ X & \xrightarrow{\theta} & F \vee X. \end{array}$$

It is well known [7] that if $f : A \to B$ is a map, then the induced cofibration $B \to C_f \to \Sigma A$ via f is a weak H'-cofibration.

A space Y is called [8,9] *co-T-space* if there is a map, co-T-structure, $\theta : Y \to Y \vee \Omega\Sigma Y$ such that $j\theta \sim (1 \times e')\Delta$, where $j : Y \vee \Omega\Sigma Y \to Y \times \Omega\Sigma Y$ is the inclusion. Clearly, any co-H-space is a co-T-space.

Definition 2.1. *A cofibration $B \xrightarrow{q} X \xrightarrow{p} F$ is called a weak co-T-cofibration if there exists a map $\theta : X \to F \vee \Omega\Sigma X$ and a homotopy $H_t : X \to F \times \Omega\Sigma X$ such that*

$$\begin{array}{ccc} B & \xrightarrow{i_2} & F \vee B \\ (a)\, q \big\downarrow & & \big\downarrow (1\vee\Omega\Sigma q e') \\ X & \xrightarrow{\theta} & F \vee \Omega\Sigma X, \end{array}$$

is homotopy commutative, where i_2 is the injection into the second factor.

(b) $H_0 = j\theta$ and $H_1 = (p \times e')\Delta$, where $j : F \vee \Omega\Sigma X \to F \times \Omega\Sigma X$ is the inclusion

Definition 2.2. *Let Y be a co-T-space with co-T-structure $\mu : Y \to Y \vee \Omega\Sigma Y$ and let $B \xrightarrow{q} X \xrightarrow{p} F$ be a weak co-T-cofibration. Then a map $f : Y \to X$ is called to be coprimitive with respect to co-T-structure μ if*

$$\begin{array}{ccc} Y & \xrightarrow{\mu} & Y \vee \Omega\Sigma Y \\ f \big\downarrow & & \big\downarrow (pf\vee\Omega\Sigma f) \\ X & \xrightarrow{\theta} & F \vee \Omega\Sigma X. \end{array}$$

Proposition 2.3. *If a cofibration $B \xrightarrow{q} X \xrightarrow{p} F$ is a weak H'-cofibration, then $B \xrightarrow{q} X \xrightarrow{p} F$ is a weak co-T-cofibration.*

Proof. Since a cofibration $B \xrightarrow{q} X \xrightarrow{p} F$ is a weak H'-cofibration, there exists a map $\theta : X \to F \vee X$ and a homotopy $H_t : X \to F \times X$ such that $(1 \vee q)i_2 \sim \theta q : B \to F \vee X$ and $H_0 = i\theta$ and $H_1 = (p \times 1)\Delta$, where i_2 is the injection into the second factor and $j : F \vee X \to F \times X$ is the inclusion. Let $\theta' = (1 \vee e')\theta : X \to F \vee \Omega\Sigma X$. Then we have, from the fact $(1 \vee \Omega\Sigma q)(1 \vee e') \sim (1 \vee e')(1 \vee q)$, that $(1 \vee \Omega\Sigma(q)e')i_2 \sim \theta' q : B \to F \vee \Omega\Sigma X$. On the other hand, let $H'_t = (1 \times e')H_t : X \to F \times \Omega\Sigma X$. Then we have that $H'_0 = j\theta'$ and $H'_1 = (p \times e')\Delta$, where $j : F \vee \Omega\Sigma X \to F \times \Omega\Sigma X$ is the inclusion. Thus we know that $B \xrightarrow{q} X \xrightarrow{p} F$ is a weak co-T-cofibration. □

Proposition 2.4. *Let Y be a co-H-space with co-H-structure $\mu : Y \to Y \vee Y$ and let $B \xrightarrow{q} X \xrightarrow{p} F$ be a weak H'-cofibration. If $f : Y \to X$ is coprimitive with respect to co-H-structure $\mu : Y \to Y \vee Y$, then $f : Y \to X$ is coprimitive with respect to co-T-structure $\mu' = (1 \vee e')\mu : Y \to Y \vee \Omega\Sigma Y$.*

Proof. Clearly we know that Y is a co-T-space with co-T-structure $\mu' = (1 \vee e')\mu Y \to Y \vee \Omega\Sigma Y$. Moreover, we know, from Proposition 2.3, that $B \xrightarrow{q} X \xrightarrow{p} F$ is a weak co-T-cofibration. Thus we have that $(pf \vee \Omega\Sigma f)\mu' \sim \theta' f : Y \to F \vee \Omega\Sigma X$. Thus we know that $f : Y \to X$ is coprimitive with respect to co-T-structure $\mu' : Y \to Y \vee \Omega\Sigma Y$. □

Theorem 2.5. *Let an (inclusion) cofibration $B \xrightarrow{q} X \xrightarrow{p} F$ be obtained by applying the suspension functor Σ to an (inclusion) cofibration $B' \xrightarrow{q'} X' \xrightarrow{p'} F'$. Then there is the homology decomposition $\{X_n, F_n, i_n, j_n, p_n, q_n\}$ for $B \xrightarrow{q} X \xrightarrow{p} F$ such that $B \xrightarrow{q_n} X_n \xrightarrow{p_n} F_n$ is a weak co-T-cofibration for each n.*

Proof. In [7] Remark 5, the homology decomposition for $B \xrightarrow{q} X \xrightarrow{p} F$ may be obtained by applying the suspension functor Σ to an (inclusion) cofibration $B' \xrightarrow{q'} X' \xrightarrow{p'} F'$, i.e, $X_n = \Sigma X'_{n-1}, F_n = \Sigma F'_{n-1}, q_n = \Sigma q'_{n-1}$ and $p_n = \Sigma p'_{n-1}$. Now we define a map $\theta : X_n \to F_n \vee \Omega\Sigma X_n$ to be the composite $\Sigma X'_{n-1} \xrightarrow{\mu} \Sigma X'_{n-1} \vee \Sigma X'_{n-1} \xrightarrow{\Sigma p'_{n-1} \vee e'} \Sigma F'_{n-1} \vee \Omega\Sigma(\Sigma X'_{n-1})$, where μ is a comultiplication in $\Sigma X'_{n-1}$. Clearly we have the following homotopy commutative diagram.

$$\begin{tikzcd}
\Sigma B' \arrow[r, "\Sigma q'_{n-1}"] \arrow[d, "e'"'] & \Sigma X'_{n-1} \arrow[d, "\Sigma e'"'] \\
\Omega\Sigma(\Sigma B') \arrow[r, "\Sigma\Omega(\Sigma q'_{n-1})"] & \Omega\Sigma(\Sigma X'_{n-1}).
\end{tikzcd}$$

From the above fact and the definition of θ, we have the following homotopy commutative diagram;

$$\begin{tikzcd}
\Sigma B' \arrow[r, "i_2"] \arrow[d, "\Sigma q'_{n-1}"'] & \Sigma F'_{n-1} \vee \Sigma B' \arrow[d, "(1 \vee \Omega\Sigma(\Sigma q'_{n-1})e')"'] \\
\Sigma X'_{n-1} \arrow[r, "\theta"] & \Sigma F'_{n-1} \vee \Omega\Sigma(\Sigma X'_{n-1}).
\end{tikzcd}$$

Thus condition (a) in 2.1 is satisfied.

Next we consider the diagram;

$$\begin{tikzcd}
\Sigma X'_{n-1} \arrow[r, "\mu"] \arrow[dr, "\Delta"] & \Sigma X'_{n-1} \vee \Sigma X'_{n-1} \arrow[r, "\Sigma p'_{n-1} \vee e'"] \arrow[d, "j"] & \Sigma F'_{n-1} \vee \Omega\Sigma(\Sigma X'_{n-1}) \arrow[d, "j"] \\
& \Sigma X'_{n-1} \times \Sigma X'_{n-1} \arrow[r, "\Sigma p'_{n-1} \times e'"] & \Sigma F'_{n-1} \times \Omega\Sigma(\Sigma X'_{n-1}).
\end{tikzcd}$$

Since $j\mu \sim \Delta$, we have $j\theta = j(\Sigma p'_{n-1} \vee e')\mu \sim (\Sigma p'_{n-1} \times e')\Delta = (p_{n-1} \times e')\Delta$. Thus condition (b) in 2.1 is satisfied. $\square$

THEOREM 2.6. *Let $B \xrightarrow{q} X \xrightarrow{p} F$ be a weak co-T-cofibration, Y a co-T-space with co-T-structure μ, $f : Y \to X$ coprimitive, and $X \xrightarrow{i} C_f \xrightarrow{\pi} \Sigma Y$ an induced cofibration via f. Then $B \xrightarrow{iq} C_f \xrightarrow{\bar{p}} C_{pf}$ is a weak co-T-cofibration.*

Proof. From [7] Lemma 2.1, $B \xrightarrow{iq} C_f \xrightarrow{\bar{p}} C_{pf}$ is a cofibration and so it suffices to show that conditions (a) and (b) in 2.1 are satisfied. By the hypothesis $B \xrightarrow{q} X \xrightarrow{p} F$ is a weak co-T-cofibration and hence there exists a map $\theta : X \to F \vee \Omega\Sigma X$ and a homotopy $H_t : X \to F \times \Omega\Sigma X$ such that $(1 \vee \Omega\Sigma q e')i_2 \sim \theta q : B \to F \times \Omega\Sigma X$, where i_2 is the injection into the second factor and $H_0 = j\theta$ and $H_1 = (p \times e')\Delta$ respectively, where $j : F \vee X \to F \times X$ is the inclusion. First we consider a composite map $\phi : X \xrightarrow{\theta} F \vee \Omega\Sigma X \xrightarrow{s \vee \Omega\Sigma i} C_{pf} \vee \Omega\Sigma C_f$, where s and i are inclusion maps. Since $phif \sim (s \vee \Omega\Sigma i)\theta f \sim (s \vee \Omega\Sigma i)(pf \vee \Omega\Sigma f)\mu = (spf \vee \Omega\Sigma i f)\mu = (\bar{p}k\iota \vee \Omega\Sigma k\iota)\mu \sim *$ and $\iota : Y \to cY$ is a cofibration, there exists a homotopy $\omega_t : cY \to C_{pf} \vee \Omega\Sigma C_f$ such that $\omega_1\iota = (s \vee \Omega\Sigma i)\theta f$ and $\omega_0 = *$. Now we define a map $\lambda : C_f \to C_{pf} \vee \Omega\Sigma C_f$ by $\lambda(y,t) = \omega_1(y,t)$, $(y,t) \in cY$, $\lambda(x) = (s \vee \Omega\Sigma i)\theta(x)$, $x \in X$. Since $\lambda(y,1) = \omega_1(y,1) = \omega_1\iota(y) = (s \vee \Omega\Sigma i)\theta f(y) = \lambda(fy)$, λ is well defined. Next we consider the following homotopy commutative diagram;

$$\begin{tikzcd}
 & X \arrow[rr, "\Delta"] \arrow[dl, "\theta"'] \arrow[dd, "i"] & & X \times X \arrow[dl, "p \times e'"'] \arrow[dd, "i \times i"] \\
F \vee \Omega\Sigma X \arrow[rr, "j"'] \arrow[dd, "s \vee \Omega\Sigma i"'] & & F \times \Omega\Sigma X \arrow[dd, "s \times \Omega\Sigma i"'] & \\
 & C_f \arrow[rr, "\Delta"] \arrow[dl, "\lambda"] & & C_f \times C_f \arrow[dl, "\bar{p} \times e'"] \\
C_{pf} \vee \Omega\Sigma C_f \arrow[rr, "j"'] & & C_{pf} \times \Omega\Sigma C_f &
\end{tikzcd}$$

Then $j\lambda i = j(s \vee \Omega\Sigma i)\theta = (s \times \Omega\Sigma i)j\theta \sim (s \times \Omega\Sigma i)(p \times e')\Delta = (\bar{p} \times e')(i \times i)\Delta = (\bar{p} \times e')\Delta i$. Since $X \xrightarrow{i} C_f \xrightarrow{q} \Sigma Y$ is an induced cofibration [7], it follows from [6, Lemma 2.2] that there exists a map $\omega : \Sigma Y \to C_{pf} \times \Omega\Sigma C_f$ such that $\nabla(\omega \vee j\lambda)\psi \sim (\bar{p} \times e')\Delta$, where $\psi : C_f \to \Sigma Y \vee C_f$

is a cooperation in the induced cofibration $X \xrightarrow{i} C_f \xrightarrow{q} \Sigma Y$, that is,

$$\begin{array}{ccc} X & \xrightarrow{i_2} & \Sigma Y \vee X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle (1\vee i)} \\ C_f & \xrightarrow{\psi} & \Sigma Y \vee C_f \\ \downarrow & & \downarrow \\ \Sigma Y & \xrightarrow{\bar{\mu}} & \Sigma Y \vee \Sigma Y \end{array}$$

and $\nabla : C_{pf} \times \Omega\Sigma C_f \vee C_{pf} \times \Omega\Sigma C_f \to C_{pf} \times \Omega\Sigma C_f$ is the folding map. Let

$$p_1 : C_{pf} \times \Omega\Sigma C_f \to C_{pf},\ p_2 : C_{pf} \times \Omega\Sigma C_f \to \Omega\Sigma C_f$$

be the projections and

$$\bar{\mu} : \Sigma Y \to \Sigma Y \vee \Sigma Y$$

the comultiplication for ΣY. Then we have $j(p_1\omega \vee p_2\omega)\bar{\mu} \sim (p_1\omega \times p_2\omega)\Delta = \omega$. Let

$$\kappa = (p_1\omega \vee p_2\omega)\bar{\mu} : \Sigma Y \to C_{pf} \vee \Omega\Sigma C_f$$

and

$$\phi = \nabla(\kappa \vee \lambda)\psi : C_f \to C_{pf} \vee \Omega\Sigma C_f.$$

Then we have $j\phi = j\nabla(\kappa \vee \lambda)\psi \sim \nabla(j\kappa \vee j\lambda)\psi \sim \nabla(\omega \jmath\lambda)\psi \sim (\bar{p} \times e')\Delta : C_f \to C_{pf} \times \Omega\Sigma C_f$. Thus the condition (b) in 2.1 holds. On the other hand, for each $b \in B$, we have $\phi iq(b) = \nabla(\kappa \vee \lambda)\psi iq(b) \sim \nabla(\kappa \vee \lambda)(1 \vee i)i_2q(b) = \nabla(\kappa \vee \lambda)(1 \vee i)(*, iq(b)) \sim \lambda iq(b) = (s \vee \Omega\Sigma i)\theta(q(b)) \sim (s \vee \Omega\Sigma i)(1 \vee \Omega\Sigma qe')i_2(b) = (s \vee \Omega\Sigma(iq)e')(*, b) = (*, \Omega\Sigma(iq)e'(b)) = (1 \vee \Omega\Sigma(iq)e')i_2(b)$. Thus the condition (a) in 2.1 holds. $\square$

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